

The Symmetric Biquadratic Diophantine Equations

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Abstract—In this paper Symmetric Diophantine equations of the form $4(xyzw) = n(xyz + yzw + zwx + wxy)$ AND $4(xyzw - ayzw - bxzw - cxyw - dxyz + abzw + bcwx + cdxy + dayz - abcw - bcdx - cday - dabz + abcd) = n(xyz + yzw + zwx + wxy - ayz - azw - awy - bxz - bzw - bwx - cxy - cyw - cwx - dyz - dxz - dxy + abz + abw + bcx + bcw + cdx + cdy + day + daz - abc - bcd - cda - dab)$ have been discussed where a, b, c, d and n are positive integers. Attempt has been made to obtain some integral solutions of these Diophantine equations using some congruence relations .

Keywords: Diophantine equation, Egyptian fraction, and integral solution.

1. INTRODUCTION

Erdos & Straus (1948) conjectured that for all integers $n \geq 2$, the rational number $\frac{4}{n}$ can be expressed as the sum of three unit fractions. The conjecture states that for every integer $n \geq 4$ there exists positive integers x, y and z so that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

These unit fractions give an Egyptian fraction representation of the number $\frac{4}{n}$. For example for $n=1801$, there exists solution $x=451$, $y=295364$ and $z=3249004$. The restriction on x, y and z to be positive make the problem difficult to solve. Computer searches have verified the truth of the conjecture upto $n \leq 10^{14}$ but proving it for all values of n remains still an open problem.

The above relation could also be written as $4xyz = n(xy + yz + zx)$. As a polynomial equation with integer variables, this is a Diophantine equation. For values of n satisfies certain congruence relations, one can find an expansion for $4/n$ automatically as an instance of a polynomial identity. For instance, whenever $n \equiv 2 \pmod{3}$, the expansion of $4/n$ is given as

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{(n-2)/3+1} + \frac{1}{n((n-2)/3+1)}.$$

Here each of the three denominators n , $(n-2)/3+1$, and $n((n-2)/3+1)$ is a polynomial of n , and each is an integer whenever $n \equiv 2 \pmod{3}$.

Jaroma (2004) presented a three term solution with one negative term given by

$$\frac{4}{n} = \frac{1}{(n-1)/2} + \frac{1}{(n+1)/2} - \frac{1}{n(n-1)(n+1)/4}.$$

Kishan.,Hari., et al. (2011) discussed the Diophantine equations of the form $3xy = n(x+y)$ and $3xyz = n(xy + yz + zx)$ etc.

Kishan.,Hari., et al. (2014) discussed the Diophantine equations of the form $3(xy - bx - ay + ab) = n(x + y - a - b)$ and $3(xyz - ayz - bxz - cxy + abz + bcx + acy - abc) = n(xy + yz + zx - bx - ay - cy - bz - az - cx + ab + bc + ac)$ etc .

In this chapter, The Symmetric biquadratic diophantine equations of the form $4(xyzw)=n(xyz+yzw+zwx+wxy)$ and $4(xyzw - ayzw - bxzw - cxyw - dxyz + abzw + bcwx + cdxy + dayz - abcw - bcdx - cday - dabz + abcd) = n(xyz + yzw + zwx + wxy - ayz - azw - awy - bxz - bzw - bwx - cxy - cyw - cwx - dyz - dxz - dxy + abz + abw + bcx + bcw + cdx + cdy + day + daz - abc - bcd - cda - dab)$ etc. have been discussed where a,b,c,d and n are positive integers. Attempts have been made to obtain some integral solutions of these Diophantine equations.

2. ANALYSIS

(i) Biquadratic Diophantine equation of the form $4(xyzw)=n(xyz+yzw+zwx+wxy)$:

The given Diophantine equation can be written as

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}. \quad (1)$$

The left hand side of the above equation can be written as

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{(n-1)}{2}+1\right)}. \quad (2)$$

Comparing equations (1) and (2), we get $x = n y = n$, $z = \frac{(n-1)}{2} + 1$ and $w = n\left(\frac{(n-1)}{2} + 1\right)$. Now if

$n \equiv 1 \pmod{2}$ then x, y, z and w are positive integers. Thus the solutions of the above Diophantine equation are:

Some solutions of the above Diophantine equation are as shown in table 1.

Table 1.

n	x	y	z	w
1	1	1	1	1
3	3	3	2	6
5	5	5	3	15
7	7	7	4	28
9	9	9	5	45
11	11	11	6	66
13	13	13	7	91
15	15	15	8	120
17	17	17	9	153

(ii) Biquadratic Diophantine equation of the form $5(xyzw)=n(xyz+yzw+zwx+wxy)$:

The given Diophantine equation can be written as

$$\frac{5}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \quad (3)$$

The left hand side of the above equation can be written as

$$\frac{5}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-2)}{3} + 1} + \frac{1}{n\left(\frac{(n-2)}{3} + 1\right)} \quad (4)$$

Comparing equations (3) and (4), we get $x = n, y = n, z = \frac{(n-2)}{3} + 1, w = n\left(\frac{(n-2)}{3} + 1\right)$. Now if

$n \equiv 2 \pmod{3}$ then x, y, z, w are positive integers. The solutions of the above Diophantine equation are:

Some solutions of the above Diophantine equation are as shown in table 2.

Table 2.

n	x	y	z	w
5	5	5	2	10
8	8	8	3	24
11	11	11	4	44
14	14	14	5	70
17	17	17	6	102
20	20	20	7	140
23	23	23	8	184
26	26	26	9	234
29	29	29	10	290

(iii) Biquadratic Diophantine equation of the form $6(xyzw)=n(xyz+yzw+zwx+wxy)$

The given Diophantine equation can be written as

$$\frac{6}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \quad \dots (5)$$

The left hand side of the above equation can be written as

$$\frac{6}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-3)}{4} + 1} + \frac{1}{n\left(\frac{(n-3)}{4} + 1\right)} \quad (6)$$

Comparing equations (5) and (6), we get $x = n, y = n, z = \frac{(n-3)}{4} + 1$ and $w = n\left(\frac{(n-3)}{4} + 1\right)$. Now if

$n \equiv 3 \pmod{4}$ then x, y, z, w are positive integers. The solutions of the above Diophantine equation are shown in table 3.

Table 3.

n	x	y	z	w
7	7	7	2	14
11	11	11	3	33
15	15	15	4	60
19	19	19	5	95
23	23	23	6	138
27	27	27	7	189
31	31	31	8	248
35	\	35	9	315

(iv) Biquadratic Diophantine equation of the form $7(xyzw)=n(xyz+yzw+zwx+wxy)$

The given Diophantine equation can be written as

$$\frac{7}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \quad \dots (7)$$

The left hand side of the above equation can be written as

$$\frac{7}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-4)}{5} + 1} + \frac{1}{n\left(\frac{(n-4)}{5} + 1\right)} \quad (8)$$

Comparing equations (7) and (8), we get $x = n, y = n, z = \frac{(n-4)}{5} + 1$ and $w = n\left(\frac{(n-4)}{5} + 1\right)$. Now if

$n \equiv 4 \pmod{5}$ then x, y, z, w are positive integers. The solutions of the above Diophantine equation areas shown in table 4.

Table 4.

n	x	y	z	w
4	4	4	1	4
9	9	9	2	18
14	14	14	3	42
19	19	19	4	76
24	24	24	5	120
29	29	29	6	174
34	34	34	7	238

(v) Biquadratic Diophantine equation of the form $8(xyzw)=n(xyz+yzw+zwx+wxy)$

The given Diophantine equation can be written as

$$\frac{8}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \quad .(9)$$

The left hand side of the above equation can be written as

$$\frac{8}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\left(\frac{n-5}{6}+1\right)} + \frac{1}{n\left(\frac{(n-5)}{6}+1\right)} \quad .(10)$$

Comparing equations (9) and (10), we get $x=n$, $y=n$, $z=\frac{(n-5)}{6}+1$ and $w=n\left(\frac{(n-5)}{6}+1\right)$. Now if

$n \equiv 5(\text{mod}6)$ then x, y, z, w are positive integers. The solutions of the above Diophantine equation are as shown in table 5.

Table 5.

n	x	y	z	w
5	5	5	1	5
11	11	11	2	22
17	17	17	3	51
23	23	23	4	92
29	29	29	5	145
35	35	35	6	210
41	41	41	7	287

(vi) Biquadratic Diophantine equation of the form

$$4(xyzw-ayzw-bxzw-cxyw-dxyz+abzw+bcwx+caxy+dayz-abcw-bcdx-cday-dabz+abcd)=n(xyz+yzw+zwx+wxy-ayz-azw-awy-bxz-bzw-bwx-cxy-cyw-cwx-dyz-dzx-dxy+abz+abw+bcx+bcw+cdx+cdy+day+daz-abc-bcd-cda-dab)$$

The given Diophantine equation can be written as

$$\frac{4}{n} = \frac{1}{x-a} + \frac{1}{y-b} + \frac{1}{z-c} + \frac{1}{w-d} \quad .(11)$$

The left hand side of the above equation can be written as

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\left(\frac{(n-2)}{3}+1\right)} + \frac{1}{n\left(\frac{(n-2)}{3}+1\right)} \quad .(12)$$

Comparing equations (11) and (12), we get $x=n+a$, $y=n+b$, $z=\frac{(n-2)}{3}+1+c$ and $w=n\left(\frac{(n-2)}{3}+1\right)+d$.

Now if $n \equiv 2(\text{mod}3)$ then x, y, z and w are positive integers.

Some solutions of the above Diophantine equation are as shown in table 6.

Table 6.

n	x	y	z	w
2	2+a	2+b	1+c	2+d
5	5+a	5+b	2+c	10+d
8	8+a	8+b	3+c	24+d
11	11+a	11+b	4+c	44+d
14	14+a	14+b	5+c	70+d
17	17+a	17+b	6+c	102+d
20	20+a	20+b	7+c	140+d

(vii) Biquadratic Diophantine equation of the form

$$5(xyzw-ayzw-bxzw-cxyw-dxyz+abzw+bcwx+caxy+dayz-abcw-bcdx-cday-dabz+abcd)=n(xyz+yzw+zwx+wxy-ayz-azw-awy-bxz-bzw-bwx-cxy-cyw-cwx-dyz-dzx-dxy+abz+abw+bcx+bcw+cdx+cdy+day+daz-abc-bcd-cda-dab)$$

The given Diophantine equation can be written as

$$\frac{5}{n} = \frac{1}{x-a} + \frac{1}{y-b} + \frac{1}{z-c} + \frac{1}{w-d} \quad .(13)$$

The left hand side of the above equation can be written as

$$\frac{5}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\left(\frac{(n-3)}{4}+1\right)} + \frac{1}{n\left(\frac{(n-3)}{4}+1\right)} \quad .(14)$$

Comparing equations (13) and (14), we get $x=n+a$, $y=n+b$, $z=\frac{(n-3)}{4}+1+c$ and $w=n\left(\frac{(n-3)}{4}+1\right)+d$.

Now if $n \equiv 3(\text{mod}4)$ then x, y, z and w are positive integers.

Some solutions of the above Diophantine equation are as shown in table 7.

Table 7.

n	x	y	z	w
3	3+a	3+b	1+c	3+d
7	7+a	7+b	2+c	14+d
11	11+a	11+b	3+c	33+d
15	15+a	15+b	4+c	60+d
19	19+a	19+b	5+c	95+d

23	23+a	23+b	6+c	138+d
27	27+a	27+b	7+c	189+d

(viii) Biquadratic Diophantine equation of the form

$$6(xyzw - ayzw - bxzw - cxyw - dxyz + abzw + bcwx + cdxy + dayz - abcw - bcdx - cday - dabz + abcd) = n(xyz + yzw + zwx + wxy - ayz - azw - awy - bxz - bzw - bwx - cxy - cyw - cwx - dyz - dzx - dxy + abz + abw + bcx + bcw + cdx + cdy + day + daz - abc - bcd - cda - dab)$$

The given Diophantine equation can be written as

$$\frac{6}{n} = \frac{1}{x-a} + \frac{1}{y-b} + \frac{1}{z-c} + \frac{1}{w-d} \quad (15)$$

The left hand side of the above equation can be written as

$$\frac{6}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-4)}{5} + 1} + \frac{1}{n\left(\frac{(n-4)}{5} + 1\right)} \quad (16)$$

Comparing equations (9) and (10), we get $x = n + a$, $y = n + b$, $z = \frac{(n-4)}{5} + 1 + c$ and $w = n\left(\frac{(n-4)}{5} + 1\right) + d$. Now if $n \equiv 4 \pmod{5}$ then x, y, z and w are positive integers.

Some solutions of the above Diophantine equation are as shown in table 8.

Table 8.

n	x	y	z	w
4	4+a	4+b	1+c	4+d
9	9+a	9+b	2+c	18+d
14	14+a	14+b	3+c	42+d
19	19+a	19+b	4+c	76+d
24	24+a	24+b	5+c	120+d
29	29+a	29+b	6+c	174+d
34	34+a	34+b	7+c	238+d

(ix) Biquadratic Diophantine equation of the form

$$7(xyzw - ayzw - bxzw - cxyw - dxyz + abzw + bcwx + cdxy + dayz - abcw - bcdx - cday - dabz + abcd) = n(xyz + yzw + zwx + wxy - ayz - azw - awy - bxz - bzw - bwx - cxy - cyw - cwx - dyz - dzx - dxy + abz + abw + bcx + bcw + cdx + cdy + day + daz - abc - bcd - cda - dab)$$

The given Diophantine equation can be written as

$$\frac{7}{n} = \frac{1}{x-a} + \frac{1}{y-b} + \frac{1}{z-c} + \frac{1}{w-d} \quad (17)$$

The left hand side of the above equation can be written as

$$\frac{7}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-5)}{6} + 1} + \frac{1}{n\left(\frac{(n-5)}{6} + 1\right)} \quad (18)$$

Comparing equations (17) and (18), we get $x = n + a$, $y = n + b$, $z = \frac{(n-5)}{6} + 1 + c$ and $w = n\left(\frac{(n-5)}{6} + 1\right) + d$.

Now if $n \equiv 5 \pmod{6}$ then x, y, z and w are positive integers.

Some solutions of the above Diophantine equation are as shown in table 9.

Table 9.

n	X	y	z	w
5	5+a	5+b	1+c	5+d
11	11+a	11+b	2+c	22+d
17	17+a	17+b	3+c	51+d
23	23+a	23+b	4+c	92+d
29	29+a	29+b	5+c	145+d
35	35+a	35+b	6+c	210+d
41	41+a	41+b	7+c	287+d

(x) Biquadratic Diophantine equation of the form

$$8(xyzw - ayzw - bxzw - cxyw - dxyz + abzw + bcwx + cdxy + dayz - abcw - bcdx - cday - dabz + abcd) = n(xyz + yzw + zwx + wxy - ayz - azw - awy - bxz - bzw - bwx - cxy - cyw - cwx - dyz - dzx - dxy + abz + abw + bcx + bcw + cdx + cdy + day + daz - abc - bcd - cda - dab)$$

The given Diophantine equation can be written as

$$\frac{8}{n} = \frac{1}{x-a} + \frac{1}{y-b} + \frac{1}{z-c} + \frac{1}{w-d} \quad (19)$$

The left hand side of the above equation can be written as

$$\frac{8}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{\frac{(n-6)}{7} + 1} + \frac{1}{n\left(\frac{(n-6)}{7} + 1\right)} \quad (20)$$

Comparing equations (19) and (20), we get $x = n + a$, $y = n + b$, $z = \frac{(n-6)}{7} + 1 + c$ and $w = n\left(\frac{(n-6)}{7} + 1\right) + d$.

Now if $n \equiv 6 \pmod{7}$ then x, y, z and w are positive integers. Some solutions of the above Diophantine equation are as shown in table 10.

Table 10.

<i>n</i>	<i>x</i>	<i>y</i>	<i>z</i>	<i>w</i>
6	6+a	6+b	1+c	6+d
13	13+a	13+b	2+c	26+d
20	20+a	20+b	3+c	60+d
27	27+a	27+b	4+c	108+d
34	34+a	34+b	5+c	170+d
41	41+a	41+b	6+c	246+d
48	48+a	48+b	7+c	336+d

3. CONCLUDING REMARKS:

In this chapter the symmetric biquadratic Diophantine equation of form $4(xyzw)=n(xyz+yzw+zwx+wxy)$ AND $4(xyzw-ayzw-bxzw-cxyw-dxyz+abzw+bcwx+cdxy+dayz-abcw-bcdx-cday-dabz+abcd)=n(xyz+yzw+zwx+wxy-ayz-azw-awy-bxz-bzw-bwx-cxy-cyw-cwx-dyz-dzx-dxy+abz+abw+bcx+bcw+cdx+cdy+day+daz-abc-bcd-cda-dab)$ have been discussed.

The solutions have been shown in the respective tables. Many solutions of these symmetric Diophantine equations and other higher order Diophantine equations can be obtained in a similar way.

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